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Solutions for games with general coalitional structure and choice sets[1]

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Abstract

In this paper we introduce the concept of quasi-building set that may underlie the coalitional structure of a cooperative game with restricted communication between the players. Each feasible coalition, including the set of all players, contains a nonempty subset called the choice set of the coalition. Only players that are in the choice set of a coalition are able to join to feasible subcoalitions to form the coalition and to obtain a marginal contribution. We demonstrate that all restricted communication systems that have been studied in the literature take the form of a quasi-building set for an appropriate set system and choice set. Every quasi-building set determines a nonempty collection of maximal strictly nested sets and each such set induces a rooted tree satisfying that every node of the tree is a player that is in the choice set of the feasible coalition that consists of himself and all his successors in the tree. Each tree corresponds to a marginal vector of the underlying game at which each player gets as payoff his marginal contribution when he joins his successors in the tree. As solution concept of a quasi-building set game we propose the average marginal vector (AMV) value, being the average of the marginal vectors that correspond to the trees induced by all maximal strictly nested sets of the quasi-building set. Properties of this solution are also studied. To establish core stability we introduce appropriate convexity conditions of the game with respect to the underlying quasi-building set. For some specifications of quasi-building sets, the AMV-value coincides with solutions known in the literature, for example, for building set games the solution coincides with the gravity center solution and the Shapley value recently defined for this class. For graph games it therefore differs from the well-known Myerson value. For a full communication system the solution coincides with the classical Shapley value.

Key words: Set system, nested set, rooted tree, chain, core, convexity, marginal vector, Shapley value

AMS subject classification: 47H10, 49J40, 52C40, 90C30, 91B50.
JEL code: C71.

1 Introduction

In the classical model of cooperative games with transferable utility (TU games) it is assumed that any subset of players can form a coalition and obtain some worth. In many situations of cooperations, however, there are restrictions for forming feasible coalitions. One of the most well-known examples is the Myerson communication graph game ([18]), where the possibility of forming coalitions is modeled by means of an undirected connected communication graph. The vertices of the graph are identified with the players, and players that are connected by an edge are able to communicate. A feasible coalition is in this model a subset of players satisfying that the induced subgraph on this subset is connected. Another example is a game on a partially ordered set, or poset, see [9, 13]. In such a game there is a partial order on the players set, for example a hierarchy, and a coalition is feasible if it is a subset of players which form an ideal, or filter, that is all players which are dominated by one of the players, or dominate that player, have to be members of the coalition. An extension of games on posets is games on augmenting systems, introduced in [4], where there is an anti-exchange closure operator on the set of players, and a coalition is feasible only if it is a closed set with respect to this closure operator. Other set systems of feasible coalitions for cooperative games with restricted coalitions that have been considered in the literature are union stable structures ([1]), convex geometries ([6, 5]), and partition system ([3]), or building sets ([17]). In all these models, marginal vectors are defined, although using different methods, and Shapley-type values are studied as solutions, that is, the average of marginal vectors is taken as solution concept to determine how much payoff every player will get.

In a cooperative game a marginal vector is a payoff vector at which with respect to some specific collection of coalitions a player receives as payoff the difference in worth when he joins to the unique smallest (with respect to set-inclusion) coalition in the collection that contains this player. For example, when all coalitions are feasible, every permutation on the set of players induces a marginal vector where each player receives his marginal contribution when he joins to the set of predecessors in that permutation. For a TU game the Shapley value is the average of the marginal vectors over all permutations. In case of a communication graph game Myerson [18] introduced the Myerson value, being the Shapley value of the restricted game for which the worth of a coalition which is not connected is restricted to be equal to the sum of the worths of the maximally connected subcoalitions, or components, of the coalition.

Instead of defining a restricted game, other Shapley-type of solution concepts that have been introduced in the literature either restrict the collection of permutations or consider the collection of maximal strictly nested sets induced by the set system. For example, in posets, augmenting systems and convex geometries only those permutations are considered for which the coalition consisting of any player with his predecessors, or successors, is feasible. A strictly nested set of a set system of feasible coalitions is a subcollection of feasible coalitions, including the set of all players, such that any two of its elements are disjoint or one is a subset of the other, and, moreover, the union of any collection of subsets of disjoint elements of the collection is not a feasible coalition. Each maximal strictly nested set induces a (rooted) tree, being a partial ordering on the set of players, and to each such a tree a marginal vector corresponds at which a player receives his marginal contribution when joining his successors in the tree. Considering maximal strictly nested sets instead of permutations for games on building set systems is used in [17] to define the gravity center, or GC-solution. Let us remark that implicitly maximal strictly nested sets were used in [15] to define the average tree solution for communication graph games and in [12] to define the Shapley value for union stable structures.

In [16] cooperative games are considered where the cooperation is restricted by a directed graph. In a directed graph one player may dominate another player in the sense that the first player is a predecessor of the other player but not reversely, while for two other players it can happen that each one is a successor of the other. An example of the latter case is the directed circular graph with three players, in which player 2 is an immediate successor of player 1, player 3 of player 2, and player 1 of player 3. For such games in [16] a collection of strictly nested sets is introduced being consistent with the underlying graph, in the sense that if in the underlying directed graph a player dominates another player, then the maximal strictly nested sets are such that the former player dominates the last player in the corresponding trees. This domination of players over other players cannot be always expressed by a set system alone. In the example of a circular directed graph game with three players the set system of connected subsets is the collection of all subsets of the set of players, but the same system is also the set system that underlies the TU game on the complete undirected graph.

In this paper we introduce the concept of quasi-building set as set system that may underly the coalitional structure of a cooperative game with restricted communication between the players. Each feasible coalition, including the set of all players, contains a nonempty subset called the choice set of the coalition. Only players that are in the choice set of a coalition are able to join to other feasible coalitions to form the coalition and are able to obtain a marginal contribution. Moreover, to have a quasi-building set, the set of remaining players of the coalition has a unique maximal partition into one or more feasible subcoalitions and any union of nonempty subsets of two or more partition members is not a feasible coalition. If the choice set of each admissible coalition coincides with itself, then we obtain a building set system. For the circular graph game example above, every player is in the choice set of the set of all three players, but the choice set for the coalition consisting of players 1 and 2 contains only player 1, the one of players 2 and 3 only contains player 2, and the one for players 1 and 3 only contains player 3. In general, for a directed graph the choice set of a connected subset contains all players in the coalition that are not being dominated within the coalition by any other player in the coalition. For an undirected graph, including the complete graph, every player of a feasible coalition belongs to the choice set of the coalition.

Every quasi-building set determines a nonempty collection of maximal strictly nested sets and each such set induces a rooted tree satisfying that every node of the tree is a player that is in the choice set of the feasible coalition that consists of himself and all his successors in the tree. Each tree corresponds to a marginal vector of the underlying game at which each player gets as payoff his marginal contribution when he joins his successors in the tree. As solution concept of a quasi-building set game we propose the average marginal vector value, being the average of the marginal vectors that correspond to the trees induced by all maximal strictly nested sets of the quasi-building set. The average marginal vector value satisfies efficiency, linearity, the null player property, the inessential coalition property and the closed coalition property. The fourth property says that the solution should not change if an inessential coalition becomes infeasible, where a feasible coalition of a quasi-building set is said to be inessential if it is not a member of any partitioning in the set system or if it is a member then of a maximal partition of a set that is obtained after deleting a player in the choice set of a feasible coalition that is also inessential. The closed coalition property says that any non-inessential feasible coalition whose players are not in the choice set of any non-inessential coalition that contains the coalition should obtain its worth. The idea is that a closed coalition can not contribute other than its own worth.

We also consider several subclasses of quasi-building sets and give for each of them convexity-type of conditions under which the average marginal vector value is an element of the core and therefore cannot be blocked by any feasible coalition. Weakly union-closed quasi-building sets are quasi-building sets that are weakly union closed, in the sense that the union of two non-disjoint feasible coalitions is also feasible if at least one of the intersecting players is in the choice set of one or both coalitions. For such a quasi-building set the convexity condition only has to hold for any pair of union-closed coalitions. For posets ([13]) and directed or undirected graphs ([18, 16]), the collection of connected sets of players with as choice set the set of undominated players within the set is a weakly union-closed quasi-building set. Mixed or directed graphs can not always be represented by set systems alone, in particular not when the graph contains directed cycles. Also set systems like partition systems ([3]), or building sets ([17]), augmenting systems ([5]) and antimatroids ([10]) induce weakly union-closed quasi-building sets, where the choice set of a feasible coalition is just the set of players for which the set of remaining players can be uniquely maximal partitioned.

Further, the class of intersection-closed quasi-building sets is studied. It holds for an intersection-closed quasi-building set that if the intersection of two feasible coalitions is nonempty, then this intersection is also a feasible coalition and, moreover, if a player is in the choice set of some feasible coalition then this player must be in the choice set of any feasible subcoalition that also contains the player. The latter property is related to the independence of irrelevant alternatives in bargaining solutions and is called the heredity property, Chernoff property, or α-axiom of Sen. For this class of quasi-building sets the convexity condition only has to hold for any pair of strongly union-closed coalitions. For a convex geometry as set system it holds that the induced quasi-building set defined in the same way as for augmenting systems is an intersection-closed quasi-building set. However, if for both these set systems the choice set of a feasible coalition is restricted to those players for which the remaining players in the set form a single feasible coalition, a chain quasi-building set is obtained, the third class of quasi-building sets we discuss. For a chain quasi-building set it holds for every player in the choice set of a feasible coalition that the set of remaining players is also a feasible coalition. All maximal strictly nested sets of a chain quasi-building set are maximal chains and reversely. In case convex geometries or augmenting systems are described in this way by chain quasi-building sets the average marginal vector value coincides with the Shapley value in [5] and [7].

The paper is organized as follows. In Section 2 quasi-building sets are introduced. In Section 3 the average marginal vector value for games on quasi-building sets is defined and properties of the new value are discussed. Subclasses of quasi-building sets and their core stability are studied in Section 4 and special cases of quasi-building sets are treated in Section 5.

2 Quasi-building sets

Let $[n] = \{1, \ldots, n\}$ be a finite set for some fixed integer n, $n \geq 2$. A set system on $[n]$ is a subset of $2^{[n]}$. A quasi-building set on $[n]$ is defined as follows.

Definition 2.1 A pair $\mathcal{Q} = (\mathcal{H}, U)$ is a *quasi-building set* on $[n]$ if it satisfies the following conditions:

(Q1) $\mathcal{H}$ is a set system on $[n]$ containing both $\emptyset$ and $[n]$ and $U : \mathcal{H} \to 2^{[n]}$ is a choice function, that is for every $H \in \mathcal{H} \setminus \{\emptyset\}$ it holds that $U(H) \neq \emptyset$ and $U(H) \subseteq H$.

(Q2) For every $H \in \mathcal{H}$ and $h \in U(H)$, there exists a unique maximal partition $P(H \setminus \{h\}) = \{S_1, \ldots, S_k\}$ of the set $H \setminus \{h\}$ for some $k \geq 1$ satisfying $S_j \in \mathcal{H}$ for $j =$

$1, \dots, k$, and for every $J \subseteq \{1, \dots, k\}$ with $|J| \geq 2$ this partition satisfies $\cup_{j \in J} T_j \notin \mathcal{H}$ for any nonempty $T_j \subseteq S_j$, $j \in J$.

A quasi-building set consists of a set system and a choice function on the set system, a mapping that assigns to every nonempty feasible set of the set system a choice set, being a nonempty subset of the set (condition (Q1)). Condition (Q2) is a kind of consistency, saying that for every feasible set of the set system and element in its choice set there exists a unique maximal partition of the remaining elements of the set into members of the set system, and no union of subsets of the sets in the partition is a member of the set system. Notice that for a given set system there may exist several choice functions which satisfy condition (Q2). One can easily check that for a given set system $\mathcal{H}$ on $[n]$ the collection of feasible choice functions is stable under union, where the union $U_1 \cup U_2$ for two choice functions U_1 and U_2 is defined by $(U_1 \cup U_2)(H) = U_1(H) \cup U_2(H)$, $H \in \mathcal{H}$.

Given a set system $\mathcal{H}$, let $\mathbf{1}(H) = H$ for any $H \in \mathcal{H}$ be the identical choice function. It is shown that partition systems, or building sets, a class of set systems introduced in [3, 17], is the only class of set systems that takes the form of quasi-building sets with the identical choice function. A set system $\mathcal{H}$ is a building set on $[n]$ if it satisfies the following conditions:

(B1) $\mathcal{H}$ is a set system on $[n]$ containing both $\emptyset$ and $[n]$.

(B2) If S, $T \in \mathcal{H}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{H}$.

(B3) For all $i \in [n]$, $\{i\} \in \mathcal{H}$.

Proposition 2.2 $(\mathcal{H}, \mathbf{1})$ *is a quasi-building set on* $[n]$ *if and only if* $\mathcal{H}$ *is a building set on* $[n]$.

Proof. Suppose $(\mathcal{H}, \mathbf{1})$ is a quasi-building set on $[n]$. (B1) obviously holds. Let S, $T \in \mathcal{H}$ with $S \cap T \neq \emptyset$. If $S \cup T = [n]$, then (B2) is verified. Suppose $S \cup T \neq [n]$. Let $j \in [n] \setminus (S \cup T)$. Then, because of (Q2) and $j \in U([n])$ since $U = \mathbf{1}$, $S \cup T$ is contained in a single member of the partition $P([n] \setminus \{j\})$. Let R be this set. If $S \cup T = R$ then (B2) is verified. Otherwise take $j' \in R \setminus (S \cup T)$, then again by (Q2) and $U = \mathbf{1}$, we get that $S \cup T$ belongs to a single member of the partition $P(R \setminus \{j'\})$, and so on. At some step, we get $S \cup T \in \mathcal{H}$ and (B2) is verified. For verifying (B3), take any $i \in [n]$ and $j \in [n] \setminus \{i\}$. Then there is some $S \in P([n] \setminus \{j\})$ containing i and take $j' \in S \setminus \{i\}$. Then take a member of $P(S \setminus \{j'\})$ containing i, and so on, until we get $\{i\} \in \mathcal{H}$.

For the reverse implication, let $\mathcal{H}$ be a building set and consider $(\mathcal{H}, \mathbf{1})$. Condition (Q1) comes from condition (B1) and the supposition $U(H) = H$. For condition (Q2), it is to show that there exists a unique maximal feasible partition of $H \setminus \{h\}$ for any

$H \in \mathcal{H}$ and $h \in H$. Take any $H \in \mathcal{H}$. Due to (B3), there is a feasible partition of $H \setminus \{h\}$ for any $h \in H$, and therefore there is a maximal one. If $\{\{i\} \mid i \in [n]\}$ is a maximal partition, then it must be unique. Suppose a maximal feasible partition of $H \setminus \{h\}$, denoted as the set of feasible coalitions $\mathcal{S}$, is not unique. Then there is another maximal feasible partition, say, $\mathcal{T}$, of $H \setminus \{h\}$. Since $\mathcal{S} \neq \mathcal{T}$, there exists an $S \in \mathcal{S}$ such that $S \nsubseteq T$ for all $T \in \mathcal{T}$. Otherwise $\mathcal{S}$ cannot be a maximal partition. Now consider a subset $\mathcal{T}^S$ of $\mathcal{T}$, $\mathcal{T}^S = \{T \in \mathcal{T} \mid T \cap S \neq \varnothing\}$. Then it follows from (B2) that $S \cup (\bigcup_{T \in \mathcal{T}^S} T) = \bigcup_{T \in \mathcal{T}^S} T \in \mathcal{H}$, which is a contradiction and therefore the maximal partition is unique. Finally, it is to see that any union of nonempty subsets of elements in the maximal partition can not be feasible. Let $P(H \setminus \{h\}) = \{S_1, \ldots, S_k\}$ be the unique maximal partition of $H \setminus \{h\}$, and suppose there exists $J \subseteq \{1, \ldots, k\}$ with $|J| \geq 2$ and some nonempty $T_j \subseteq S_j$, $j \in J$, such that $T = \cup_{j \in J} T_j \in \mathcal{H}$. Then from (B2), it must hold that $(\cup_{j \in J} S_j) \cup T \in \mathcal{H}$, since $T \cap S_j \neq \emptyset$ for all $j \in J$, which contradicts that $\{S_1, \ldots, S_k\}$ is a maximal partition. □

An important property of a quasi-building set is that it contains maximal strictly nested sets. For a quasi-building set a strictly nested set is a generalization of nested sets for building sets, see [8],[17],[19], and is defined as follows.

Definition 2.3 A set system $\mathcal{N}$ on $[n]$ is a *strictly nested set* in a quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$ if it satisfies the following conditions:
(N1) For any different S, $T \in \mathcal{N}$ either $S \subset T$ or $T \subset S$ or $S \cap T = \emptyset$.
(N2) For any collection of k, $k \geq 2$, disjoint nonempty subsets $T_1, \ldots, T_k$ in $\mathcal{N}$ it holds that

$$T'_1 \cup \cdots \cup T'_k \notin \mathcal{H}$$

for any nonempty $T'_j \subseteq T_j$, $j = 1, \ldots, k$.
(N3) For any $h \in [n]$ there exists H in $\mathcal{N}$ such that $h \in U(H)$ and H is a minimal set in $\mathcal{N}$ containing h.
(N4) $\mathcal{N}$ contains $[n]$.

Condition (N1) says that any two different sets of a strictly nested set in a quasi-building set are either disjoint or one is a subset of the other. A set satisfying this property is also known in the literature as *laminar* or *hierarchy*, see, for example, [14]. Condition (N2) gives the strength of the nested property, see, for example, [8]. It says that the union of subsets of disjoint sets of a strictly nested set cannot be a member of the underlying set system. Condition (N3) says that for any element in $[n]$ there exists a minimal set in the strictly nested set such that this element is in the choice set of this set. Together with the first condition this minimal set is unique. Condition (N4) reqires that any strictly nested set contains $[n]$.

A strictly nested set $\mathcal{N}$ in a quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$ is *maximal* if it contains n different nonempty sets of the set system $\mathcal{H}$. To every maximal strictly nested set $\mathcal{N}$ in $\mathcal{Q}$ there corresponds a (rooted) tree $T^{\mathcal{N}}$ with vertex set $[n]$. Given a maximal strictly nested set $\mathcal{N}$ and element $i \in [n]$, let the set $F^{\mathcal{N}}(i)$ be the unique minimal element of $\mathcal{N}$ containing i, then in the tree $T^{\mathcal{N}}$ the set $F^{\mathcal{N}}(i)$ will be the set of subordinates of i including i and element $j \in [n]$ belongs to the set $S^{\mathcal{N}}(i)$ of successors of i if $F^{\mathcal{N}}(j)$ is a maximal subset of $F^{\mathcal{N}}(i) \setminus \{i\}$ in $\mathcal{N}$. A tree $T^{\mathcal{N}}$ describes how the set $[n]$ can be constructed by letting its nodes join to their sets of subordinates. Any element $i \in [n]$ constructs the feasible set $F^{\mathcal{N}}(i)$ in the maximal strictly nested set $\mathcal{N}$ by joining to all the feasible sets $F^{\mathcal{N}}(j)$, $j \in S^{\mathcal{N}}(i)$, that are constructed by his successors in the tree. These latter sets form a maximal partition of the set of subordinates of i (Q2) and satisfy that no union of subsets is a feasible set of the underlying set system (N2), i.e., these sets or any of their subsets are not able to construct a feasible set without i. The root of the tree compiles the whole set $[n]$, also being a feasible set (Q1), by joining to all feasible sets that are constructed by its successors. An element is allowed to construct a feasible set only if it belongs to the choice set of that set. The collection of maximal strictly nested sets in a quasi-building set describes all different possibilities in which the set $[n]$ can be compiled in this way. The collection of maximal strictly nested sets of a quasi-building set $\mathcal{Q}$ on $[n]$ is denoted by $\mathcal{M}(\mathcal{Q})$. The next theorem shows that any quasi-building set contains at least one maximal strictly nested set.

Theorem 2.4 *Let $\mathcal{Q}$ be a quasi-building set on $[n]$, then $\mathcal{M}(\mathcal{Q}) \neq \emptyset$.*

Proof. Let $\mathcal{Q} = (\mathcal{H}, U)$. We first construct a tree T on the vertex set $[n]$ as follows. From (Q1) it follows that $[n] \in \mathcal{H}$ and $U([n]) \neq \emptyset$. As root of T we take any element $r \in U([n])$. According to (Q2) there exists a unique maximal partition $S_1, \ldots, S_k$ of $[n] \setminus \{r\}$ for some $k \geq 1$ such that $S_j \in \mathcal{H}$ for all $j = 1, \ldots, k$. In each S_j, $j = 1, \ldots, k$, there exists according to (Q1) an element $r_j \in U(S_j)$, which we connect to the root r, i.e., in T each node r_j is a successor of the root r, $j = 1, \ldots, k$. Again, for each $j = 1, \ldots, k$, by (Q2) there exists a unique maximal partition $S_{j,1}, \ldots, S_{j,k_j}$ of $S_j \setminus \{r_j\}$ such that $S_{j,h} \in \mathcal{H}$ and by (Q1) $U(S_{j,h}) \neq \emptyset$ for $h = 1, \ldots, k_j$, and we let r_j be the root of the subtree of T for the restriction of the quasi-building set $\mathcal{Q}$ on the set S_j, and so on. In this way, we obtain a rooted tree T on $[n]$ directed from the root. The collection of sets of subordinates $F(i)$, $i \in [n]$, being the principal ideals of T, form a maximal strictly nested set $\mathcal{N}$ in $\mathcal{Q}$, i.e., $\mathcal{N} = \{F(i) |\ i \in [n]\}$. $\square$

The next two examples show that a same set system with different choice functions may lead to different sets of maximally strictly nested sets.

Example 2.5 Consider the quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$ on $\{1, 2, 3\}$ where $\mathcal{H} = 2^{[n]}$ and $U(H) = H$ for all $H \in \mathcal{H}$. There are six maximal strictly nested sets, corresponding to all six line-trees on $[n]$.

Example 2.6 Consider the quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$ on $\{1, 2, 3\}$ where $\mathcal{H} = 2^{[n]}$ and $U(\{i\}) = \{i\}$, for $i = 1, 2, 3$, $U(\{1, 2\}) = \{1\}$, $U(\{1, 3\}) = \{3\}$, $U(\{2, 3\}) = \{2\}$, $U([n]) = [n]$. There are three maximal strictly nested sets, $\mathcal{N}_1 = \{\{1\}, \{1, 3\}, \{1, 2, 3\}\}$, $\mathcal{N}_2 = \{\{2\}, \{1, 2\}, \{1, 2, 3\}\}$, and $\mathcal{N}_3 = \{\{3\}, \{2, 3\}, \{1, 2, 3\}\}$, where $\mathcal{N}_1$ corresponds to the line-tree $T^{\mathcal{N}_1} = \{(2, 3), (3, 1)\}$ having element 2 as root, $\mathcal{N}_2$ corresponds to the line-tree $T^{\mathcal{N}_2} = \{(3, 1), (1, 2)\}$ having element 3 as root, and $\mathcal{N}_3$ corresponds to the line-tree $T^{\mathcal{N}_3} = \{(1, 2), (2, 3)\}$ having element 1 as root. Note that $\{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is not a maximal strictly nested set because $2 \notin U(\{1, 2\})$.

3 The average marginal vector value

Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building set on $[n]$ and $v : \mathcal{H} \to \mathbb{R}$ a function such that $v(\emptyset) = 0$. We consider $\mathcal{H}$ as a coalition structure on a set of n players and v as a characteristic function of a cooperative game with $v(H)$, $H \in \mathcal{H}$, the worth of feasible coalition H. In the remaining of this paper we identify a cooperative game with its characteristic function. The pair $(v, \mathcal{Q})$ is a *quasi-building set game* on $[n]$. The collection of all quasi-building set games on $[n]$ is denoted by $\mathcal{V}$. A *value* is a mapping f from $\mathcal{V}$ to $\mathbb{R}^{[n]}$, assigning the payoff vector $f(v, \mathcal{Q})$ to any game $(v, \mathcal{Q}) \in \mathcal{V}$.

As solution concept for quasi-building set games we propose the average of the marginal vectors of all maximal strictly nested sets in the quasi-building set. Given a quasi-building set game $(v, \mathcal{Q}) \in \mathcal{V}$, for a maximal strictly nested set $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$ the *marginal vector* $m^{\mathcal{N}}(v, \mathcal{Q}) \in \mathbb{R}^{[n]}$ is defined by

$$m_i^{\mathcal{N}}(v, \mathcal{Q}) = v(F^{\mathcal{N}}(i)) - \sum_{j \in S^{\mathcal{N}}(i)} v(F^{\mathcal{N}}(j)), \; i \in [n],$$

where, for any $h \in [n]$, $F^{\mathcal{N}}(h)$ is the set of subordinates of h including h and $S^{\mathcal{N}}(h)$ is the set of successors of h in the tree $T^{\mathcal{N}}$ corresponding to $\mathcal{N}$. At a marginal vector of a maximal strictly nested set every player receives as payoff what he contributes when he joins his subordinates in the corresponding tree.

Definition 3.1 On the class of quasi-building set games $\mathcal{V}$ the *Average Marginal Vector Value*, or *AMV-value*, assigns to every quasi-building set game $(v, \mathcal{Q}) \in \mathcal{V}$ the payoff vector

$$AMV(v, \mathcal{Q}) = \frac{1}{|\mathcal{M}(\mathcal{Q})|} \sum_{\mathcal{N} \in \mathcal{M}(\mathcal{Q})} m^{\mathcal{N}}(v, \mathcal{Q}).$$

The AMV-value of a quasi-building set game is the average of the marginal vectors induced by all maximal strictly nested sets in the quasi-building set. The AMV-value is well defined on the class of quasi-building set games, since according to Theorem 2.4 every quasi-building set has at least one maximal strictly nested set.

We next discuss some properties of the AMV-value. The first two properties are standard.

Definition 3.2 A value f on $\mathcal{V}$ satisfies *efficiency* if for all $(v, \mathcal{Q}) \in \mathcal{V}$ it holds that

$$\sum_{i \in [n]} f_i(v, \mathcal{Q}) = v([n]).$$

An efficient value generates for any quasi-building set game a payoff vector which allocates the worth of the grand coalition among players.

Proposition 3.3 *The AMV-value satisfies efficiency.*

Proof. Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a quasi-building set game. It suffices to show that the marginal vector of any maximal strictly nested set of $\mathcal{Q}$ is efficient, since the AMV-value is the average of all such vectors. For a maximal strictly nested set $\mathcal{N}$ and $i \in [n]$ it holds that

$$\begin{aligned}
\sum_{j \in F^{\mathcal{N}}(i)} m_j^{\mathcal{N}}(v, \mathcal{Q}) &= \sum_{j \in F^{\mathcal{N}}(i)} v(F^{\mathcal{N}}(j)) - \sum_{j \in F^{\mathcal{N}}(i)} \sum_{k \in S^{\mathcal{N}}(j)} v(F^{\mathcal{N}}(k)) \\
&= \sum_{j \in F^{\mathcal{N}}(i)} v(F^{\mathcal{N}}(j)) - \sum_{j \in F^{\mathcal{N}}(i) \setminus \{i\}} v(F^{\mathcal{N}}(j)) \\
&= v(F^{\mathcal{N}}(i)),
\end{aligned}$$

since for each $j \in F^{\mathcal{N}}(i)$, $j \neq i$, there is precisely one element $k \in F^{\mathcal{N}}(i)$ such that k is a successor of j in the corresponding tree $T^{\mathcal{N}}$. From (N4) it holds that $[n] \in \mathcal{N}$ and there exists $r \in [n]$ such that $F^{\mathcal{N}}(r) = [n]$, where r is the root of $T^{\mathcal{N}}$. Therefore,

$$\sum_{i \in [n]} m_i^{\mathcal{N}}(v, \mathcal{Q}) = \sum_{i \in F^{\mathcal{N}}(r)} m_i^{\mathcal{N}}(v, \mathcal{Q}) = v(F^{\mathcal{N}}(r)) = v([n]).$$

□

Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building set system on $[n]$. For any two quasi-building set games $(v, \mathcal{Q})$ and $(w, \mathcal{Q})$ in $\mathcal{V}$ and $a, b \in \mathbb{R}$ the quasi-building set game $(av + bw, \mathcal{Q})$ is defined by $(av + bw)(H) = av(H) + bw(H)$ for all $H \in \mathcal{H}$.

Definition 3.4 A value f on $\mathcal{V}$ satisfies *linearity* if for any quasi-building set games $(v, \mathcal{Q})$ and $(w, \mathcal{Q})$ in $\mathcal{V}$ and $a, b \in \mathbb{R}$ it holds that

$$f(av + bw, \mathcal{Q}) = af(v, \mathcal{Q}) + bf(w, \mathcal{Q}).$$

Proposition 3.5 *The AMV-value satisfies linearity.*

Proof. Let $(v, \mathcal{Q})$ and $(w, \mathcal{Q})$ be two quasi-building set games a quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$, let $a, b \in \mathbb{R}$ be given, and consider the quasi-building set game $(av + bw, \mathcal{Q})$. Since those three games are on the same quasi-building set, the collection of all maximal strictly nested sets, $\mathcal{M}(\mathcal{Q})$, is identical for them. Clearly, for each $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$ and $i \in [n]$ it holds that

$$\begin{aligned} m_i^{\mathcal{N}}(av + bw, \mathcal{Q}) &= (av + bw)(F^{\mathcal{N}}(i)) - \sum_{j \in S^{\mathcal{N}}(i)} (av + bw)(F^{\mathcal{N}}(j)) \\ &= a(v(F^{\mathcal{N}}(i)) - \sum_{j \in S^{\mathcal{N}}(i)} v(F^{\mathcal{N}}(j))) + b(w(F^{\mathcal{N}}(i)) - \sum_{j \in S^{\mathcal{N}}(i)} w(F^{\mathcal{N}}(j))) \\ &= am_i^{\mathcal{N}}(v, \mathcal{Q}) + bm_i^{\mathcal{N}}(w, \mathcal{Q}). \end{aligned}$$

$\square$

Since maximal strictly nested sets consist of feasible coalitions that satisfy particular properties, there may exist coalitions that are feasible but are not an element of any maximal strictly nested set. Such a coalition is inessential in the sense that neither feasibility nor its worth affects the AMV-value.

Definition 3.6 Given a quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$, a coalition $H \in \mathcal{H}$ is *inessential* if $H \in \cup_{k=1}^{n} \mathcal{I}^k$, where $\mathcal{I}^n, \ldots, \mathcal{I}^1$ are recursively defined as follows:

1. $\mathcal{I}^n = \emptyset$;

2. For $k = n - 1, \ldots, 1$, $H \in \mathcal{I}^k$ if $|H| = k$ and for every $T \in \mathcal{H} \setminus (\cup_{h=k+1}^{n} \mathcal{I}^h)$ such that $H \subset T$ it holds that $H \notin P(T \setminus \{i\})$ for all $i \in U(T)$.

An inessential coalition of a quasi-building set is a feasible coalition that can be only a member of a maximal partition of a set which is obtained after deleting a player in the choice set from an inessential coalition. The collection of inessential coalitions is defined recursively, since subsets of an inessential coalition may not be inessential.

Example 3.7 Consider the quasi-building set $\mathcal{Q}(\mathcal{G}) = (\mathcal{H}, U)$ on $\{1, 2\}$ where $\mathcal{H} = \{\{1\}, \{2\}, \{1, 2\}\}$ and $U(\{1\}) = \{1\}$, $U(\{2\}) = \{2\}$, $U(\{1, 2\}) = \{1\}$. There is one maximal strictly nested set $\mathcal{N} = \{\{2\}, \{1, 2\}\}$. The singleton $\{1\}$ is an inessential coalition, since $2 \notin U(\{1, 2\})$.

The main property of an inessential coalition is that it does not show up in any maximal strictly nested set of the underlying quasi-building set. Let $\mathcal{I}(\mathcal{Q})$ denote the collection of inessential coalitions of a quasi-building set $\mathcal{Q}$.

Lemma 3.8 *Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building set on $[n]$ and $H \in \mathcal{H}$. Then $H \notin \mathcal{N}$ for any maximal strictly nested set $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$ if and only if $H \in \mathcal{I}(\mathcal{Q})$.*

Proof. Suppose $H \in \mathcal{I}(\mathcal{Q})$ and H is an element of some maximal strictly nested set $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$. Then from conditions (N1) and (N4) it holds that there exists a unique minimal $H_1 \in \mathcal{N}$ containing H. From (N3) and because $H \in \mathcal{N}$, there exists $h_1 \in H_1 \setminus H$ with $h_1 \in U(H_1)$ such that $H \in P(H_1 \setminus \{h_1\})$, which would be a contradiction unless $H_1 \in \mathcal{I}(\mathcal{Q})$. If $H_1 \in \mathcal{I}(\mathcal{Q})$, by following the same argument there exists $H_2 \in \mathcal{N}, H_2 \supset H_1$, and some $h_2 \in U(H_2)$ with $H_1 \in P(H_2 \setminus \{h_2\})$. Then it must hold that $H_2 \in \mathcal{I}(\mathcal{Q})$ to avoid the contradiction, and so on. Since the player set is finite and $[n] \notin \mathcal{I}(\mathcal{Q})$, we obtain a finite sequence of feasible coalitions $(H_1, \dots, H_m)$ for some $m < n$ satisfying $H_1 \subset \cdots \subset H_m$, $H_{m-1} \in \mathcal{I}(\mathcal{Q})$ and $H_m \notin \mathcal{I}(\mathcal{Q})$, whereas $H_{m-1} \in P(H_m \setminus \{h_m\})$ for some $h_m \in U(H_m)$, which is a contradiction.

Next, suppose $H \notin \mathcal{I}(\mathcal{Q})$, then there exists $H_1 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ such that $H \in P(H_1 \setminus \{h_1\})$ for some $h_1 \in U(H_1)$. Since H_1 is not inessential, there exists $H_2 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ such that $H_1 \in P(H_2 \setminus \{h_2\})$ for some $h_2 \in U(H_2)$, and so on. Since the player set is finite, there is a finite sequence of players $(h_1, \dots, h_m)$ and feasible sets $(H_1, \dots, H_m)$ for some $m < n$ such that $H_m = [n]$ and $H_{j-1} \in P(H_j \setminus \{h_j\})$ for $j = 1, \dots, m$. Then as done in the proof of Theorem 2.4 we may construct a tree T on $[n]$ having player h_m as root and containing (h_j, h_{j-1}) for $j = 2, \dots, m$ as directed links. Then T corresponds to a strictly maximal nested set $\mathcal{N}$ of $\mathcal{Q}$ with $H \in \mathcal{N}$. $\square$

An inessential coalition of a quasi-building set is a feasible coalition that does not appear in any maximal partition and therefore an inessential coalition is not an element of any maximal strictly nested set. The inessential coalition property of a quasi-building set is defined as follows.

Definition 3.9 A value f on the class of quasi-building set games $\mathcal{V}$ satisfies the *inessential coalition property* if for every $(v, \mathcal{Q}) \in \mathcal{V}$ with $\mathcal{Q} = (\mathcal{H}, U)$ and any inessential coalition $I \in \mathcal{I}(\mathcal{Q})$ it holds that $f(v, \mathcal{Q}) = f(v, (\mathcal{H} \setminus \{I\}, U))$.

This property states that an allocation is independent of feasibility of any inessential coalition and therefore is also independent of the worths of inessential coalitions. Therefore we have the next corollary.

Corollary 3.10 *A value f on the class of quasi-building set games $\mathcal{V}$ satisfies the inessential coalition property if and only if for any $(v, \mathcal{Q}), (w, \mathcal{Q}) \in \mathcal{V}$ such that $v(H) = w(H)$ for all $H \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, it holds that $f(v, \mathcal{Q}) = f(w, \mathcal{Q})$.*

The AMV-value on the class of quasi-building set games satisfies this property, since according to Lemma 3.8 the collection of maximal strictly nested sets of a quasi-building set does not change by deleting or adding inessential coalitions or changing their worths.

Proposition 3.11 *The AMV-value satisfies the inessential coalition property.*

While an inessential coalition never shows up in any maximal strictly nested set, there might also be feasible coalitions that are members of every maximal strictly nested set. Such a coalition is called a closed coalition of the quasi-building set.

Definition 3.12 Given a quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$, a coalition $H \in \mathcal{H}$ is a *closed coalition* if for every $T \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $H \subset T$ it holds for all $i \in U(T)$ that there exists $S \in P(T \setminus \{i\})$ such that $H \subseteq S$.

A closed coalition of a quasi-building set is a feasible coalition whose players are not selected by the choice function of any non-inessential feasible coalition that contains the set. Notice that the grand coalition $[n]$ is by definition a closed coalition. Since members of a closed coalition can never contribute to other coalitions their total payoff should be equal to the worth of the coalition.

Definition 3.13 A value f on the class of quasi-building set games $\mathcal{V}$ satisfies the *closed coalition property* if for every $(v, \mathcal{Q}) \in \mathcal{V}$ with $\mathcal{Q} = (\mathcal{H}, U)$ and closed coalition $H \in \mathcal{H}$ it holds that $\sum_{i \in H} f_i(v, \mathcal{Q}) = v(H)$.

An allocation that satisfies the closed coalition property gives as total payoff to the players who form a closed coalition exactly the worth of the coalition. Since the grand coalition is a closed coalition, this property implies efficiency.

Proposition 3.14 *The AMV-value satisfies the closed coalition property.*

Proof. Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a quasi-building set game with $\mathcal{Q} = (\mathcal{H}, U)$ and $H \in \mathcal{H}$ a closed coalition. Suppose $H \notin \mathcal{N}$ for some $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$. From condition (N3) and because $H \notin \mathcal{N}$, there exist $i \in H$ and $T \in \mathcal{N}$ such that $i \in U(T)$ and $H \subset T$. This implies that there is no $S \in P(T \setminus \{i\})$ such that $S \supseteq H$, which is a contradiction since H is a closed coalition. Since the AMV-value is the average of the marginal vectors corresponding to all maximal strictly nested sets, it suffices to show that $\sum_{j \in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) = v(H)$ for any $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$. Take any $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$. Since $H \in \mathcal{N}$, there exists $i \in H$ such that $F^{\mathcal{N}}(i) = H$. Then it follows that

$$\sum_{j \in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) = \sum_{j \in F^{\mathcal{N}}(i)} m_j^{\mathcal{N}}(v, \mathcal{Q}) = v(F^{\mathcal{N}}(i)) = v(H).$$

□

The null player property is a widely known concept. In a standard TU-game, a player is a null player if he never contributes to the worth of a coalition by joining to it. For a quasi-building set game, however, a player may not join to all coalitions due to the underlying restricted set system and choice function. A null player for a quasi-building set game is defined as follows.

Definition 3.15 Given a quasi-building set game $(v, \mathcal{Q}) \in \mathcal{V}$ with $\mathcal{Q} = (\mathcal{H}, U)$, player $i \in N$ is a *null player* if for all $H \in \mathcal{H}$ such that $i \in U(H)$ it holds that

$$v(H) - \sum_{K \in P(H \setminus \{i\})} v(K) = 0.$$

Notice that the definition of a null player depends not only on the set system but also the choice function. The definition of a null player can be weakened by restricting $H \in \mathcal{H}$ to the set of feasible coalitions that are not inessential. Null players contribute nothing and should receive zero payoff.

Definition 3.16 A value f on the class of quasi-building set games $\mathcal{V}$ satisfies the *null player property* if for every $(v, \mathcal{Q}) \in \mathcal{V}$ it holds that $f_i(v, \mathcal{Q}) = 0$ for any null player $i \in [n]$.

Proposition 3.17 *The AMV-value satisfies the null player property.*

Proof. Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a quasi-building set game with $\mathcal{Q} = (\mathcal{H}, U)$ and null player $i \in [n]$. Since the AMV-value is the average of the marginal vectors corresponding to all maximal strictly nested sets, it suffices to show that $m_i^{\mathcal{N}}(v, \mathcal{Q}) = 0$ for all $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$. Take any $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$. Since $F^{\mathcal{N}}(i)$ is the minimum set in $\mathcal{N}$ containing i, it follows from condition (N3) that $i \in U(F^{\mathcal{N}}(i))$. By condition (Q2), $P(F^{\mathcal{N}}(i) \setminus \{i\})$ is the unique partition of $F^{\mathcal{N}}(i) \setminus \{i\}$, and thus $P(F^{\mathcal{N}}(i) \setminus \{i\}) = \{F^{\mathcal{N}}(j) \mid j \in S^{\mathcal{N}}(i)\}$. Since i is a null player, it follows that

$$\begin{aligned} m_i^{\mathcal{N}}(v, \mathcal{Q}) &= v(F^{\mathcal{N}}(i)) - \sum_{j \in S^{\mathcal{N}}(i)} v(F^{\mathcal{N}}(j)) \\ &= v(F^{\mathcal{N}}(i)) - \sum_{K \in P(F^{\mathcal{N}}(i) \setminus \{i\})} v(K) = 0. \end{aligned}$$

□

4 Subclasses of quasi-building sets

In this section we discuss several special cases of quasi-building sets in relation to previous studies on TU-games with communication restriction. The situations that are covered in this section are restrictions expressed by communication graphs, directed or undirected, and by set systems such as augmenting systems and convex geometries. For such situations the AMV-value is well defined for specific quasi-building sets with appropriately defined choice functions. For each subclass, a convexity condition under which the AMV-value of the game lies in the core, will be given.

The core of a game is the set of efficient and stable payoff vectors. On the class of quasi-building set games the core is defined as follows.

Definition 4.1 Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a quasi-building set game, where $\mathcal{Q} = (\mathcal{H}, U)$, then the *core* of the game $(v, \mathcal{Q})$ is given by the set

$$C(v, \mathcal{Q}) = \{x \in \mathbb{R}^n |\ x([n]) = v([n]),\ x(H) \geq v(H),\ H \in \mathcal{H}\}.$$

The core reflects the property that only coalitions that are feasible are able to block a payoff vector.

4.1 Weakly union-closed quasi-building sets

In this subsection we consider the subclass of weakly union-closed quasi-building sets.

Definition 4.2 A pair $\mathcal{Q} = (\mathcal{H}, U)$ is a *weakly union-closed quasi-building set* on $[n]$ if $\mathcal{Q}$ is a quasi-building set and the following holds:

(Q3) For any $H_1 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $H_2 \in \mathcal{H}$ satisfying $U(H_1) \cap H_2 \neq \emptyset$ it holds that $H_1 \cup H_2 \in \mathcal{H}$.

Condition (Q3) says that the union of two sets in the set system, at least one of two is non-inessential, is also in the set system if their intersection is nonempty and some element in the intersection is contained in the choice set of a non-inessential set. The conditions together imply that in each feasible set H of the underlying set system of a quasi-building set there is a nonempty subset, which could be the set itself, for which it holds that each element in this subset is able to join in a unique way disjoint sets in the set system to form H (condition (Q2)) and, when H is also non-inessential, for each element in this subset the union of H and any other set in the set system, to which the element also belongs, is a feasible set (condition (Q3)). Note that condition (Q3) is weaker than the weak union-closedness condition for set systems. The existence of maximal strictly nested sets is not violated by requiring (Q3), since this condition demands the underlying set system to be richer. In the next section we demonstrate that weakly union closed quasi-building set games cover games with a communication graph (undirected, directed or mixed), augmenting systems, and posets.

Next we discuss conditions under which the AMV-value is stable on the class of weakly union-stable quasi-building sets.

Definition 4.3 Given a weakly union-closed quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$, a pair (A, B) of subsets of $[n]$ is *union-closed* if $A \in \mathcal{H}$ and there exists $i \in A \setminus B$ such that $B \cup \{i\} \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $i \in U(B \cup \{i\})$.

A feasible coalition A and a possibly infeasible coalition B of a weakly union-closed quasi-building set is union-closed if there exists a player in A outside B such that when this player, say player i, would join the players in B the resulting coalition $C = B \cup \{i\}$ is non-inessential and contains player i in its choice set. Note that $A \cup B \in \mathcal{H}$ because of (Q3). Player $i \in A \cap C$ is as member of the choice set of C able to connect the non-inessential coalition C to the feasible coalition A to form the feasible coalition $A \cup B$. A convexity condition for the games on this class of quasi-building set is defined on union-closed pairs of underlying quasi-building set as follows.

Definition 4.4 Let $\mathcal{Q} = (\mathcal{H}, U)$ be a weakly union-closed quasi-building set on $[n]$. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-supermodular* if for any union-closed pair (A, B) and maximal partition $\mathcal{D}$ of $A \cap B$ into sets of $\mathcal{H}$ it holds that

$$f(A) + \sum_{K \in P(B)} f(K) \leq f(A \cup B) + \sum_{K \in \mathcal{D}} f(K). \tag{1}$$

Notice that condition (Q2) implies that the set B has a unique maximal partition $P(B)$. Since the maximal partition of the intersection into feasible coalitions may not be unique, the condition should hold for all such maximal partitions. The next example shows that, for a union-closed pair, a maximal partition of its intersection into feasible coalitions may not be unique.

Example 4.5 Consider a quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$ on $\{1,2,3,4,5\}$, where $\mathcal{H} = \{\{1,2,3,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2\}, \{2,3\}, \{1\}, \ldots, \{5\}\}$ and the choice function is such that $U(\{1,2,3,4,5\}) = \{4,5\}$, $U(\{1,2,3,4\}) = \{1\}$, $U(\{1,2,3,5\}) = \{3\}$, $U(\{1,2\}) = \{1\}$, $U(\{2,3\}) = \{3\}$, $U(\{i\}) = \{i\}$, $i \in \{1,2,3,4,5\}$. This $\mathcal{Q}$ is weakly union-closed, and there are two maximal strictly nested sets, $\mathcal{N}_1 = \{\{1,2,3,4,5\}, \{1,2,3,5\}, \{1,2\}, \{2\}, \{5\}\}$ and $\mathcal{N}_2 = \{\{1,2,3,4,5\}, \{1,2,3,4\}, \{2,3\}, \{2\}, \{4\}\}$. The pair (A, B) with $A = \{1,2,3,4\}$ and $B = \{1,2,3,5\}$ is union-closed. Their intersection $A \cap B = \{1,2,3\}$ has two maximal partitions into feasible coalitions, namely $\{\{1\}, \{2,3\}\}$ and $\{\{1,2\}, \{3\}\}$.

In the next theorem it is shown that the $\mathcal{Q}$-supermodularity is a sufficient condition for the stability of the AMV-value.

Theorem 4.6 *Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a weakly union-closed quasi-building set game. If the characteristic function v is $\mathcal{Q}$-supermodular, then the core $C(v, \mathcal{Q})$ contains the AMV-value.*

Proof. We show that for every maximal strictly nested set $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$ and coalition $H \in \mathcal{H}$ it holds that

$$\sum_{j \in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) \geq v(H). \tag{2}$$

Since the AMV-value is efficient and the core is a convex set this implies that the AMV-value is in the core.

Take any $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$ and $H \in \mathcal{H}$. Let $H_1, \ldots, H_s$ be the maximal connected subsets of H in the tree $T^{\mathcal{N}}$ corresponding to $\mathcal{N}$. For $k = 1, \ldots, s$ denote $H_k = \{i_1^k, \ldots, i_{t_k}^k\}$ and let $h < l$ if $i_h^k \in F^{\mathcal{N}}(i_l^k)$. For $k = 1, \ldots, s$ denote $r_k = i_{t_k}^k$ and let $h < l$ if $r_h \in F^{\mathcal{N}}(r_l)$. Since $T^{\mathcal{N}}$ is a tree, r_k is the root of the subtree of $T^{\mathcal{N}}$ on $F^{\mathcal{N}}(r_k)$ containing the set H_k, $k = 1, \ldots, s$. Moreover, $F^{\mathcal{N}}(r_s)$ contains the set H and also $F^{\mathcal{N}}(r_k)$, $k = 1, \ldots, s-1$. Since for $k = 1, \ldots, s$ it holds that

$$H_k = F^{\mathcal{N}}(r_k) \setminus \Big(\cup_{h=1}^{t_k} \big(\cup_{j \in S^{\mathcal{N}}(i_h^k) \setminus H_k} F^{\mathcal{N}}(j) \big) \Big),$$

we obtain that

$$\sum_{j \in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(F^{\mathcal{N}}(r_k)) - \sum_{h=1}^{t_k} \sum_{j \in S^{\mathcal{N}}(i_h^k) \setminus H_k} v(F^{\mathcal{N}}(j)) \Big). \tag{3}$$

To show that the latter expression is at least or equal to $v(H)$, define $I^k = H \cup (\cup_{h=1}^{k} F^{\mathcal{N}}(r_h))$ for $k = 0, \ldots, s$ and $I_h^k = I^{k-1} \cup (\cup_{j=1}^{h} F^{\mathcal{N}}(i_j^k))$ for $h = 0, \ldots, t_k$, $k = 1, \ldots, s$. Notice that $I^0 = H$ and $I^s = F^{\mathcal{N}}(r_s)$ and that for $k = 1, \ldots, s$ it holds that $I_0^k = I^{k-1}$ and $I_{t_k}^k = I^k$. We first show by induction that $I_h^k \in \mathcal{H}$ for all $h = 0, \ldots, t_k$, $k = 1, \ldots, s$. Since $I_0^1 = I^0 = H$ it holds that $I_0^1 \in \mathcal{H}$. Suppose $I_h^1 \in \mathcal{H}$ for some $h < t_1$. Since $F^{\mathcal{N}}(i_h^1) \in \mathcal{H}$ and $i_h^1 \in H \cap U(F^{\mathcal{N}}(i_h^1)) \subseteq I_h^1 \cap U(F^{\mathcal{N}}(i_h^1))$, it follows from condition (Q3) that the union I_{h+1}^1 of the sets I_h^1 and $F^{\mathcal{N}}(i_{h+1}^1)$ is in $\mathcal{H}$. In particular, this implies for $h = t_1 - 1$ that $I_{t_1}^1$ is in $\mathcal{H}$. Since $I_{t_1}^1 = I^1 = I_0^2$, it also holds that $I_0^2 \in \mathcal{H}$. Continuing the same argument, we obtain by induction that $I_h^k \in \mathcal{H}$ for all k and h.

Let $A = I_{h-1}^k$ and $B = \cup_{j \in S^{\mathcal{N}}(i_h^k)} F^{\mathcal{N}}(j)$ for some $h = 1, \ldots, t_k$, $k = 1, \ldots, s$. Then $A \in \mathcal{H}$ and $A \cup B = I_h^k \in \mathcal{H}$. It holds that $i_h^k \in A \setminus B$, $i_h^k \in U(B \cup \{i_h^k\})$ and $B \cup \{i_h^k\} = F^{\mathcal{N}}(i_h^k) \in \mathcal{N}$, and therefore $B \cup \{i_h^k\} \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$. Hence, the pair (A, B) is union-closed in $\mathcal{Q}$. Concerning the intersection of A and B, for $j \in S^{\mathcal{N}}(i_h^k) \setminus H_k$ define

$$D_h^k(j) = \{r_i \mid F^{\mathcal{N}}(r_i) \subset F^{\mathcal{N}}(j),\ \nexists l \in \{i+1, \ldots, k-1\} \text{ with } F^{\mathcal{N}}(r_i) \subset F^{\mathcal{N}}(r_l) \subset F^{\mathcal{N}}(j)\}.$$

Then there is a partition of $A \cap B$ into sets of $\mathcal{H}$, namely

$$\mathcal{D} = \{F^{\mathcal{N}}(r_i) \mid r_i \in D_h^k(j),\ j \in S^{\mathcal{N}}(i_h^k) \setminus H_k\} \cup \{F^{\mathcal{N}}(j) \mid j \in S^{\mathcal{N}}(i_h^k) \cap H_k\}.$$

This partition is maximal, because every element of this partition is an element in the maximal strictly nested set $\mathcal{N}$ and if this partition were not maximal, condition (N2) would be violated. Therefore $\mathcal{D}$ is a maximal partition of $A \cap B$ into sets of $\mathcal{H}$ and the pair (A, B) is union-closed in $\mathcal{Q}$. Since v is $\mathcal{Q}$-supermodular, $A = I_{h-1}^k$, and $A \cup B = I_h^k$, it holds that

$$v(I^k_{h-1}) + \sum_{K \in P(B)} v(K) \leq v(I^k_h) + \sum_{K \in \mathcal{D}} v(K).$$

Since $P(B) \cap \mathcal{D} = \{F^{\mathcal{N}}(j) \mid j \in S^{\mathcal{N}}(i^k_h) \cap H_k\}$, the terms indexed by these sets cancel on both sides and we obtain

$$v(I^k_{h-1}) + \sum_{j \in S^{\mathcal{N}}(i^k_h) \setminus H_k} v(F^{\mathcal{N}}(j)) \leq v(I^k_h) + \sum_{j \in S^{\mathcal{N}}(i^k_h) \setminus H_k} \sum_{r_i \in D^k_h(j)} v(F^{\mathcal{N}}(r_i)).$$

Applying this inequality successively for $h = 1, \ldots, t_k$, $k = 1, \ldots, s$, we obtain that

$$v(I^1_0) + \sum_{k=1}^{s} \sum_{h=1}^{t_k} \sum_{j \in S^{\mathcal{N}}(i^k_h) \setminus H_k} v(F^{\mathcal{N}}(j)) \leq v(I^s_{t_s}) + \sum_{k=1}^{s} \sum_{h=1}^{t_k} \sum_{j \in S^{\mathcal{N}}(i^k_h) \setminus H_k} \sum_{r_i \in D^k_h(j)} v(F^{\mathcal{N}}(r_i)).$$

Since $I^1_0 = H$, $I^s_{t_s} = F^{\mathcal{N}}(r_s)$ and each r_i, $i = 1, \ldots, s-1$, belongs to precisely one $D^k_h(j)$ for some $j \in S^{\mathcal{N}}(i^k_h) \setminus H_k$, $h \in \{1, \ldots, t_k\}$, $k \in \{2, \ldots, s\}$, it follows that

$$\sum_{j \in H} m^{\mathcal{N}}_j(v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(F^{\mathcal{N}}(r_k)) - \sum_{h=1}^{t_k} \sum_{j \in S^{\mathcal{N}}(i^k_h) \setminus H_k} v(F^{\mathcal{N}}(j)) \Big) \geq v(H).$$

□

4.2 Intersection-closed quasi-building set

In this subsection we consider the subclass of intersection-closed quasi-building sets.

Definition 4.7 A pair $\mathcal{Q} = (\mathcal{H}, U)$ is an *intersection-closed quasi-building set* on $[n]$ if $\mathcal{Q}$ is a quasi-building set and the following holds:

(Q4) For $H_1, H_2 \in \mathcal{H}$ satisfying $H_1 \cap H_2 \neq \varnothing$, then $H_1 \cap H_2 \in \mathcal{H}$.

(Q5) For $H_1, H_2 \in \mathcal{H}$ satisfying $H_1 \subset H_2$ and $i \in U(H_2) \cap H_1$, then $i \in U(H_1)$.

Condition (Q4) reflects the name of this subclass, as it is known as intersection-closed condition, which is a particular condition convex geometries possess. Condition (Q5) states that if a player is in the choice set of a feasible coalition, then he must be in the choice set of a feasible coalition which is a subset of the coalition that contains this player. This is in line with the property called independence of irrelevant alternatives or α-axiom of Sen, or heredity axiom, as a choice for a feasible set remains a choice for any smaller feasible set, as long as this choice is available. This property may not be compatible with the union-closed quasi-building set. Regarding this respect, reconsider Example 2.6, which is an union-closed quasi-building set. For this example, (Q5) is not satisfied for $H_1 = \{1, 2\}$ and $H_2 = [n]$. In the next section we demonstrate that intersection-closed quasi-building set games cover games with convex geometries.

For the class of intersection-closed quasi-building set games, a convexity condition is defined as follows.

Definition 4.8 Let $\mathcal{Q} = (\mathcal{H}, U)$ be an intersection-closed quasi-building set on $[n]$. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-convex* if

$$f(T) - \sum_{K \in P(T \setminus \{i\})} f(K) \leq f(S) - \sum_{K \in P(S \setminus \{i\})} f(K)$$

for any $S \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q}), T \in \mathcal{H}$, $T \subset S$ and $i \in U(S) \cap T$.

A game on an intersection-closed quasi-building set is $\mathcal{Q}$-convex if the marginal loss caused by a player is greater whenever he is removed from a larger feasible and not inessential coalition. This condition is in line with a convexity condition introduced in [6] on the class of games on convex geometries.

Theorem 4.9 *Let $(v, \mathcal{Q}) \in \mathcal{V}$ be an intersection-closed quasi-building set game. If the characteristic function v is $\mathcal{Q}$-convex, then the core $C(v, \mathcal{Q})$ contains the AMV-value.*

Proof. Let $\mathcal{N}$ be a maximal nested set in $\mathcal{Q}$ and let $\mathcal{Q} = (\mathcal{H}, U)$. We show that for every $H \in \mathcal{H}$ it holds that

$$\sum_{j \in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) \geq v(H). \tag{4}$$

Denote $H = \{i_1, \ldots, i_s\}$ and let $h < l$ if $i_h \in F^{\mathcal{N}}(i_l)$ in the tree $T^{\mathcal{N}}$ corresponding to $\mathcal{N}$. Since $\mathcal{N}$ is a strictly nested set, i_s is uniquely determined. The left hand side of (4) is expressed as

$$\sum_{j \in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(F^{\mathcal{N}}(i_k)) - \sum_{K \in P(F^{\mathcal{N}}(i_k) \setminus \{i_k\})} v(K) \Big).$$

From (Q4), the set Q_k is feasible where $Q_k = F^{\mathcal{N}}(i_k) \cap H$, $k = 1, \ldots, s$, since $F^{\mathcal{N}}(i_k) \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $H \in \mathcal{H}$. It follows that $Q_s = H$ and $Q_k \subset H$ for $k = 1, \ldots, s-1$. Moreover it holds for $k = 1, \ldots, s$ that $i_k \in U(F^{\mathcal{N}}(i_k)) \cap Q_k$, $Q_k \subset F^{\mathcal{N}}(i_k)$. From (Q5) it follows for $k = 1, \ldots, s$ that $i_k \in U(Q_k)$ and thus $P(Q_k \setminus \{i_k\})$ exists. Clearly, $P(Q_k \setminus \{i_k\})$ consists of a set of Q_js with $j < k$, $k = 1, \ldots, s$. Since the game is $\mathcal{Q}$-convex we have

$$m_{i_k}^{\mathcal{N}}(v, \mathcal{Q}) = v(F^{\mathcal{N}}(i_k)) - \sum_{K \in P(F^{\mathcal{N}}(i_k) \setminus \{i_k\})} v(K) \geq v(Q_k) - \sum_{K \in P(Q_k \setminus \{i_k\})} v(K),$$

for $k = 1, \dots, s$. Adding up this inequality for $k = 1, \dots, s$, we have

$$\sum_{j\in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) \geq \sum_{k=1}^{s} \Big(v(Q_k) - \sum_{K\in P(Q_k\setminus\{i_k\})} v(K) \Big).$$

Since $Q_s = H$, the inequality becomes

$$\sum_{j\in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) \geq v(H) + \sum_{k=1}^{s-1} v(Q_k) - \sum_{k=1}^{s} \sum_{K\in P(Q_k\setminus\{i_k\})} v(K).$$

The last two terms cancel out since $\cup_{k=1}^{s} P(Q_k \setminus \{i_k\}) = \{Q_1, \dots, Q_{S-1}\}$ and the desired result follows. $\square$

With properly defined pairs of sets, strongly union-closed pairs, there is an equivalent expression to $\mathcal{Q}$-convexity. On the class of intersection-closed quasi-building set, a strongly union-closed pair is defined as follows.

Definition 4.10 Given an intersection-closed quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$, a pair (A, B) of subsets of $[n]$ is *strongly union-closed* if $A \in \mathcal{H}$, $A \cup B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, and there exists $i \in A \setminus B$ such that $B \cup \{i\} \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $i \in U(A \cup B)$.

Note that the definition is different from Definition 4.3 of a union-closed pair on the class of weakly union-closed quasi-building set. Now the condition requires that $i \in U(A \cup B)$. It then follows from (Q5) that $i \in U(B \cup \{i\})$ and $P(B)$ exists. It also excludes situations where the union $A \cup B$ is an inessential coalition. Due to the condition (Q4) it holds that $(A \cap B) \cup \{i\} = A \cap (B \cup \{i\}) \in \mathcal{H}$, and together with (Q5) it holds that $i \in U((A \cap B) \cup \{i\})$ and $P(A \cap B)$ exists. Since $i \in U(B \cup \{i\})$ does not imply $i \in U(A \cup B)$ nor $A \cup B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, for a quasi-building set which is both weakly union-closed and intersection-closed, a strongly union-closed pair is union-closed, but a union-closed pair is not necessarily strongly union-closed.

Theorem 4.11 Let $\mathcal{Q} = (\mathcal{H}, U)$ be an intersection-closed quasi-building set on $[n]$. A function $v : \mathcal{H} \to \mathbb{R}$ is $\mathcal{Q}$-convex if and only if

$$v(A) + \sum_{K\in P(B)} v(K) \leq v(A \cup B) + \sum_{K\in P(A\cap B)} v(K) \tag{5}$$

holds for any strongly union-closed pair (A, B).

Proof. For the necessity, take any $S \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q}), T \in \mathcal{H}$, $T \subset S$ and $i \in U(S) \cap T$. Since $i \in U(S)$, the partition $P(S \setminus \{i\})$ exists. From (Q5) and $i \in U(S) \cap T$, it holds that

$i \in U(T)$ and therefore $P(T \setminus \{i\})$ exists. Then with $A = T$ and $B = S \setminus \{i\}$, from (5) it follows that

$$v(T) + \sum_{K \in P(S \setminus \{i\})} v(K) \leq v(S) + \sum_{K \in P(T \setminus \{i\})} v(K),$$

because of the fact that $S \cup T = S$ and $(S \setminus \{i\}) \cap T = T \setminus \{i\}$.

For the sufficiency, take any strongly union-closed pair (A, B). The game v is $\mathcal{Q}$-convex and it holds for some $i_1 \in U(A \cup B) \cap A$ that

$$v(A \cup B) - \sum_{K \in P(A \cup B \setminus \{i_1\})} v(K) \geq v(A) - \sum_{K \in P(A \setminus \{i_1\})} v(K),$$

since $A \cup B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, $A \in \mathcal{H}$, $A \subset A \cup B$ and $U(A \cup B) \cap A \neq \emptyset$ for the pair (A, B). From (5), it is to show

$$\sum_{K \in P(A \cup B \setminus \{i_1\})} v(K) - \sum_{K \in P(A \setminus \{i_1\})} v(K) \geq \sum_{K \in P(B)} v(K) - \sum_{K \in P(A \cap B)} v(K). \qquad (6)$$

For this purpose, we show by induction that there is a sequence $(i_1, \ldots, i_k)$ of all k different elements in $A \setminus B$ such that

$$\sum_{K \in P(A \cup B \setminus \{i_1, \ldots, i_h\})} v(K) - \sum_{K \in P(A \setminus \{i_1, \ldots, i_h\})} v(K) \geq \sum_{K \in P(A \cup B \setminus \{i_1, \ldots, i_{h+1}\})} v(K) - \sum_{K \in P(A \setminus \{i_1, \ldots, i_{h+1}\})} v(K),$$

for $h = 1, \ldots, k$.

First consider the case with $h = 1$. Take any $X \in P(A \cup B \setminus \{i_1\})$ with $X \setminus B \neq \emptyset$. If there is no such X, it implies that $A \setminus B = \{i_1\}$ and therefore $A \cap B = A \setminus \{i_1\}$ and $A \cup B \setminus \{i_1\} = B$, which leads to (6). Suppose there is such X. We show that $U(X) \setminus B \neq \emptyset$. Because $B \cup \{i_1\} \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $i_1 \in U(B \cup \{i_1\})$, there exists a maximal strictly nested set $\mathcal{N}$ with $B \cup \{i_1\} \in \mathcal{N}$ and $P(B) \subseteq \mathcal{N}$. There also exists unique $\overline{X} \in \mathcal{N}$ which minimally covers X. It follows from $B \cup \{i_1\} \in \mathcal{N}$, $X \setminus B \neq \emptyset$ and $i_1 \notin X$ that $B \cup \{i_1\} \subsetneq \overline{X}$. Since $\overline{X}, B \cup \{i_1\} \in \mathcal{N}$, it holds that there exists $i_2 \in U(\overline{X}) \setminus B$ with $P(\overline{X} \setminus \{i_2\}) \subseteq \mathcal{N}$. This means that $i_2 \in X$, and thus $i_2 \in X \setminus B$, because $\overline{X}$ minimally covers X. From (Q5) it follows that $i_2 \in U(X)$ since $\overline{X} \supset X$ and $i_2 \in U(\overline{X}) \cap X$. There is unique $S \in P(A \setminus \{i_1\})$ with $i_2 \in S$ and it must hold that $S \subset X$. From (Q5), it then holds that $i_2 \in U(S)$ and therefore $P(S \setminus \{i_2\})$ exists. From $\mathcal{Q}$-convexity it holds that

$$v(X) - \sum_{K \in P(X \setminus \{i_2\})} v(K) \geq v(S) - \sum_{K \in P(S \setminus \{i_2\})} v(K),$$

since $S \subset X$, $X \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $i_2 \in U(X) \cap S$. Among the elements of $P(A \cup B \setminus \{i_1\})$, i_2 appears only in X and therefore $P(A \cup B \setminus \{i_1, i_2\}) = (P(A \cup B \setminus \{i_1\}) \setminus \{X\}) \cup P(X \setminus \{i_2\})$.

Also among the elements of $P(A\setminus\{i\})$, i_2 appears only in S and therefore $P(A\setminus\{i_1,i_2\}) = (P(A\setminus\{i_1\})\setminus\{S\})\cup P(S\setminus\{i_2\})$. It thus follows that

$$\begin{aligned}
&\sum_{K\in P(A\cup B\setminus\{i_1\})} v(K) - \sum_{K\in P(A\setminus\{i_1\})} v(K) \\
&= \sum_{K\in P(A\cup B\setminus\{i_1\})\setminus\{X\}} v(K) + v(X) - \sum_{K\in P(A\setminus\{i_1\})\setminus\{S\}} v(K) - v(S) \\
&\geq \sum_{K\in P(A\cup B\setminus\{i_1\})\setminus\{X\}} v(K) + \sum_{K\in P(X\setminus\{i_2\})} v(K) - \sum_{K\in P(A\setminus\{i_1\})\setminus\{S\}} v(K) - \sum_{K\in P(S\setminus\{i_2\})} v(K) \\
&= \sum_{K\in P(A\cup B\setminus\{i_1,i_2\})} v(K) - \sum_{K\in P(A\setminus\{i_1,i_2\})} v(K).
\end{aligned}$$

Next assume that the supposition is correct up to some h with $1 < h < k-1$. That is,

$$\sum_{K\in P(A\cup B\setminus\{i_1,\dots,i_h\})} v(K) - \sum_{K\in P(A\setminus\{i_1,\dots,i_h\})} v(K) \geq \sum_{K\in P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})} v(K) - \sum_{K\in P(A\setminus\{i_1,\dots,i_{h+1}\})} v(K).$$

Since $h < k-1$, there exists $Y \in P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})$ such that $Y\setminus B \neq \emptyset$. It is to show that there exists $i_{h+2}\in Y\setminus B$ such that $i_{h+2}\in U(Y)$. Consider a maximal strictly nested set $\mathcal{N}$ such that $B\cup\{i_1\}\in\mathcal{N}$ and $P(B)\subseteq\mathcal{N}$ and take $\overline{Y}\in\mathcal{N}$ that minimally covers Y. Since $Y\in P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})$, $Y\setminus B\neq\emptyset$ and $B\cup\{i_1\}\in\mathcal{N}$, it follows that $B\cup\{i_1\}\subsetneq\overline{Y}$. Since $\overline{Y}, B\cup\{i_1\}\in\mathcal{N}$, it holds that there exists $i_{h+2}\in U(\overline{Y})\setminus B$ with $P(\overline{Y}\setminus\{i_{h+2}\})\subseteq\mathcal{N}$. This means that $i_{h+2}\in Y$, and thus $i_{h+2}\in Y\setminus B$, because $\overline{Y}$ minimally covers Y. From (Q5) it follows that $i_{h+2}\in U(Y)$ since $\overline{Y}\supset Y$ and $i_{h+2}\in U(\overline{Y})\cap Y$. There is unique $T\in P(A\setminus\{i_1,\dots,i_{h+1}\})$ with $i_{h+2}\in T$, and it holds that $i_{h+2}\in U(T)$, because of (Q5) and $T\subset Y$. Then from $\mathcal{Q}$-convexity it holds that

$$v(Y) - \sum_{K\in P(Y\setminus\{i_{h+2}\})} v(K) \geq v(T) - \sum_{K\in P(T\setminus\{i_{h+2}\})} v(K),$$

since $T\subset Y$, $Y\in\mathcal{H}\setminus\mathcal{I}(\mathcal{Q})$ and $i_{h+2}\in U(Y)\cap T$. Among the elements of $P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})$, i_{h+2} appears only in Y and therefore $P(A\cup B\setminus\{i_1,\dots,i_{h+2}\}) = (P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})\setminus\{Y\})\cup P(Y\setminus\{i_{h+2}\})$. Also among the elements of $P(A\setminus\{i_1,\dots,i_{h+1}\})$, i_{h+2} appears only in T and therefore $P(A\setminus\{i_1,\dots,i_{h+2}\}) = (P(A\setminus\{i_1,\dots,i_{h+1}\})\setminus\{T\})\cup P(T\setminus\{i_{h+2}\})$. Therefore it follows with a similar calculation as for the case where $h=1$

that

$$
\begin{aligned}
&\sum_{K\in P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})} v(K) - \sum_{K\in P(A\setminus\{i_1,\dots,i_{h+1}\})} v(K)\\
&= \sum_{K\in P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})\setminus\{Y\}} v(K) + v(Y) - \sum_{K\in P(A\setminus\{i_1,\dots,i_{h+1}\})\setminus\{T\}} v(K) - v(T)\\
&\geq \sum_{K\in P(A\cup B\setminus\{i_1,\dots,i_{h+1}\})\setminus\{Y\}} v(K) + \sum_{K\in P(Y\setminus\{i_{h+2}\})} v(K) - \sum_{K\in P(A\setminus\{i_1,\dots,i_{h+1}\})\setminus\{T\}} v(K)\\
&\quad - \sum_{K\in P(T\setminus\{i_{h+2}\})} v(K)\\
&= \sum_{K\in P(A\cup B\setminus\{i_1,\dots,i_{h+2}\})} v(K) - \sum_{K\in P(A\setminus\{i_1,\dots,i_{h+2}\})} v(K).
\end{aligned}
$$

This concludes the induction. By using this result successively, we obtain

$$
\begin{aligned}
\sum_{K\in P(A\cup B\setminus\{i_1\})} v(K) - \sum_{K\in P(A\setminus\{i_1\})} v(K) &\geq \sum_{K\in P(A\cup B\setminus\{i_1,i_2\})} v(K) - \sum_{K\in P(A\setminus\{i_1,i_2\})} v(K)\\
&\geq \cdots \geq \sum_{K\in P(A\cup B\setminus\{i_1,\dots,i_k\})} v(K) - \sum_{K\in P(A\setminus\{i_1,\dots,i_k\})} v(K)\\
&= \sum_{K\in P(B)} v(K) - \sum_{K\in P(A\cap B)} v(K),
\end{aligned}
$$

since $A \cup B \setminus \{i_1, \dots, i_k\} = B$ and $A \setminus \{i_1, \dots, i_k\} = A \cap B$. □

From the theorem it follows that on the subclass of games on quasi-building sets that are both weakly union-closed and intersection-closed $\mathcal{Q}$-supermodularity implies $\mathcal{Q}$-convexity. The next example shows a $\mathcal{Q}$-convex weakly union-closed and intersection-closed quasi-building set game which is not $\mathcal{Q}$-supermodular.

Example 4.12 Consider a quasi-building set game $(v, \mathcal{Q})$ with $\mathcal{Q} = (\mathcal{H}, U)$ on $\{1, 2, 3, 4, 5\}$, where $\mathcal{H} = \{\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3\}, \{2, 4\}, \{1\}, \{2\}, \{3\}, \{5\}\}$, the choice function is such that $U(\{1, 2, 3, 4, 5\}) = \{4\}$, $U(\{1, 2, 3, 4\}) = \{4\}$, $U(\{1, 2, 3\}) = \{2\}$, $U(\{2, 4\}) = \{4\}$, $U(\{1\}) = \{1\}$, $U(\{2\}) = \{2\}$, $U(\{3\}) = \{3\}$, $U(\{5\}) = \{5\}$, and the characteristic function is such that $v(\{1, 2, 3, 4, 5\}) = 5$, $v(\{1, 2, 3, 4\}) = 2$, $v(\{1, 2, 3\}) = 2$, $v(\{2, 4\}) = 3$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{5\}) = 0$. Note that $\mathcal{Q}$ is both weakly union-closed and intersection-closed, with $\{1, 2, 3, 4\}$, $\{2, 4\}$ and $\{2\}$ being inessential coalitions. There is one maximal strictly nested set, $\mathcal{N} = \{\{1, 2, 3, 4, 5\}, \{1, 2, 3\}, \{1\}, \{3\}, \{5\}\}$, and $m^{\mathcal{N}}(v, \mathcal{Q}) = (0, 2, 0, 3, 0)$ is in the core of the game. This game is $\mathcal{Q}$-convex, but not $\mathcal{Q}$-supermodular (take the pair (A, B) with $A = \{2, 4\}$ and $B = \{1, 3\}$).

4.3 Chain quasi-building sets

In this subsection we consider the subclass of chain quasi-building sets.

Definition 4.13 A pair $\mathcal{Q} = (\mathcal{H}, U)$ is a *chain quasi-building set* on $[n]$ if it satisfies the following conditions:

(Q1) $\mathcal{H}$ is a set system on $[n]$ containing both $\emptyset$ and $[n]$ and $U : \mathcal{H} \to 2^{[n]}$ is a choice function, that is for every $H \in \mathcal{H} \setminus \{\emptyset\}$ it holds that $\emptyset \neq U(H) \subseteq H$.

(Q2)' For every $H \in \mathcal{H}$ and $h \in U(H)$, $H \setminus \{h\} \in \mathcal{H}$.

Condition (Q2)' comes from condition (Q2) and the one-point extension property, i.e., if $H \in \mathcal{H}, H \neq [n]$, then there exists $i \in [n] \setminus S$ such that $S \cup \{i\} \in \mathcal{H}$ and $i \in U(S \cup \{i\})$. The next lemma shows that any maximal strictly nested set of a chain quasi-building set is a maximal chain.

Lemma 4.14 *Suppose $\mathcal{Q} = (\mathcal{H}, U)$ is a chain quasi-building set on $[n]$. Then any maximal strictly nested set of $\mathcal{Q}$ is a maximal chain, i.e., it consists of n feasible coalitions $H_1, \ldots, H_n$ with $H_1 \subset \cdots \subset H_n$.*

Proof. Let $\mathcal{N}$ be a maximal strictly nested set of $\mathcal{Q}$. Suppose that $\mathcal{N}$ is not a chain and there exists $H \in \mathcal{H}$ and $h \in U(H)$ such that $P(H \setminus \{h\}) = \{T_1, \ldots, T_k\}$ with $T_l \in \mathcal{N}$ for $l = 1, \ldots, k$, for some $k \geq 2$. Then (Q2)' and $h \in U(H)$ imply that $H \setminus \{h\} \in \mathcal{H}$ and therefore $T_1 \cup \cdots \cup T_k \in \mathcal{H}$, which contradicts that $\mathcal{N}$ is a maximal strictly nested set. $\square$

For the stability of the AMV-value on the class of chain quasi-building set games, we introduce the following condition. Since every maximal strictly nested set is a chain, the condition has a flavor of chains.

Definition 4.15 Let $\mathcal{Q} = (\mathcal{H}, U)$ be a chain quasi-building set on $[n]$. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-chain-increasing* if

$$\sum_{i=1}^{k} (f(S^i) - f(S^i \setminus H_i)) \geq f(H)$$

holds for any $H \in \mathcal{H}$ and $S^1, \ldots, S^k \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $S^1 \subset \cdots \subset S^k$, $H \subset S^k$ and $S^i \setminus H_i \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, where $H_i = (S^i \setminus S^{i-1}) \cap H \neq \varnothing$, for all $i = 1, \ldots, k$.

This condition states that the worth of a coalition is less than or equal to the sum of their marginal contributions to any increasing sequence of non-inessential and feasible coalitions.

Theorem 4.16 *Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a chain quasi-building set game on $[n]$. If the characteristic function v is $\mathcal{Q}$-chain-increasing, then the core $C(v, \mathcal{Q})$ contains the AMV-value.*

Proof. Let $\mathcal{Q} = (\mathcal{H}, U)$. We show that for every $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$ and $H \in \mathcal{H}$ it holds that

$$\sum_{j\in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) \geq v(H).$$

Take any $\mathcal{N} \in \mathcal{M}(\mathcal{Q})$ and $H \in \mathcal{H}$. Let $H_1, \ldots, H_k$ be the maximal connected subsets of H in the tree $T^{\mathcal{N}}$ corresponding to $\mathcal{N}$. From Lemma 4.14, $T^{\mathcal{N}}$ is a line-tree. Therefore for each $H_i, i = 1, \ldots, k$, there exists unique $\overline{S}^i \in \mathcal{N}$ such that $H_i \subseteq \overline{S}^i$ and there is no $S' \in \mathcal{N}$ with $H_i \subseteq S' \subset \overline{S}^i$, and there exists unique $\underline{S}^i \in \mathcal{N}$ such that $H_i \cap \underline{S}^i = \varnothing$ and there is no $S' \supset \underline{S}^i, S' \in \mathcal{N}$ with $H_i \cap S' = \varnothing$. Note that $\overline{S}^i \setminus H_i = \underline{S}^i$. For each $i = 1, \ldots, k$, there exists unique $\overline{r}_i$ such that $F^{\mathcal{N}}(\overline{r}_i) = \overline{S}^i$. It holds that $\overline{r}_i \in H_i$ for $i = 1, \ldots, k$, otherwise there exists $S' \in \mathcal{N}$ such that $H_i \subseteq S' \subset \overline{S}^i$. There also exists unique $\underline{r}_i \in \underline{S}^i$ such that $F^{\mathcal{N}}(\underline{r}_i) = \underline{S}^i$ and it holds that $\underline{r}_i \notin H$ for $i = 1, \ldots, k$. Note that $H_i = F^{\mathcal{N}}(\overline{r}_i) \setminus F^{\mathcal{N}}(\underline{r}_i)$. The sets $\overline{S}^i, i = 1, \ldots, k$, can be ordered such that $\overline{S}^1 \subset \cdots \subset \overline{S}^k$ and $H \subset \overline{S}^k$. Therefore $\overline{S}^1, \ldots, \overline{S}^k$ and H satisfy the condition for $\mathcal{Q}$-chain-increasing. Note that $\overline{S}^i \notin \mathcal{I}(\mathcal{Q})$ for $i = 1, \ldots, k$ since $\overline{S}^i \in \mathcal{N}$. Also it holds that $\overline{S}^i \setminus H_i \notin \mathcal{I}(\mathcal{Q})$ for $i = 1, \ldots, k$ since $\overline{S}^i \setminus H_i = \underline{S}^i$ and $\underline{S}^i \in \mathcal{N}$. Since $T^{\mathcal{N}}$ is a line-tree and $H_i = F^{\mathcal{N}}(\overline{r}_i) \setminus F^{\mathcal{N}}(\underline{r}_i)$, as marginal contribution, feasible coalition H receives

$$\sum_{j\in H} m_j^{\mathcal{N}}(v, \mathcal{Q}) = \sum_{i=1}^{k} \sum_{j\in H_i} m_j^{\mathcal{N}}(v, \mathcal{Q}) = \sum_{i=1}^{k} (v(F^{\mathcal{N}}(\overline{r}_i)) - v(F^{\mathcal{N}}(\underline{r}_i))) = \sum_{i=1}^{k} (v(\overline{S}^i) - v(\overline{S}^i \setminus H_i)),$$

which is greater or equal to $v(H)$ since v is $\mathcal{Q}$-chain-increasing. $\square$

5 Special cases

In this section we discuss how a quasi-building set can be constructed when the underlying communication structure has some standard properties, such as a collection of connected subsets of a (directed) graph, or a combinatorial structure as augmenting systems, convex geometries, and posets. When a communication situation is represented by a directed graph, the choice function can be used to serve to represent the underlying dominance relation. In case where a communication situation is expressed by certain set systems, the choice function can be taken such that it induces precisely the collection of all maximal strictly nested sets coming from the set system. A quasi-building set coming from any of such special cases belongs to one of the subclasses studied in the previous section.

5.1 Graphical quasi-building set

A *mixed* graph $G = (V, E)$ with $E = L \cup A \subseteq \{(i, j) \mid i, j \in [n], i \neq j\}$ on $[n]$ consists of a vertex set V equal to the set $[n]$ and a set of edges E which is constituted from *links*

$L = \{(i,j) \in E \mid (j,i) \in E)\}$, undirected edges, and *arrows* $A = \{(i,j) \in E \mid (j,i) \notin E)\}$, a set of directed edges. A mixed graph without arrows is an undirected graph and a mixed graph without links is a directed graph. Given a mixed graph $G = (V,E)$ with $E = L \cup A$, the underlying undirected graph $\overline{G} = (V, \overline{E})$ is obtained by making from every arrow a link. A graph (V,E) is a complete graph on $[n]$ if $A = \emptyset$ and $L = \{(i,j) \mid i,j \in [n], i \neq j\}$.

Definition 5.1 Given a mixed graph $G = (V,E)$ with $E = L \cup A$ on $[n]$, the pair $\mathcal{Q}(G) = (\mathcal{H}, U)$ on $[n]$ consists of a set system $\mathcal{H}$ and mapping $U : H \to 2^{[n]}$ given by the following conditions:

- $\mathcal{H}$ is the set of all connected subsets of $\overline{G}$.
- U assigns to every $H \in \mathcal{H}$ a set of nodes which are undominated in the mixed subgraph $G(H)$ of G on H, being the set of nodes in H from which there is a directed path in the subgraph $G(H)$ to any of its predecessors in the set H.

Note that in case of an undirected graph all nodes in a connected subset are undominated.

Lemma 5.2 *For a connected mixed graph $G = (V,E)$ with $E = L \cup A$ on $[n]$, $\mathcal{Q}(G)$ is a weakly union-closed quasi-building set on $[n]$.*

Proof. Let $\mathcal{Q}(G) = (\mathcal{H}, U)$. Since $\overline{G}$ is connected, and by definition the empty set is connected, condition (Q1) is satisfied. For any nonempty $H \in \mathcal{H}$, which is a finite set of nodes being connected in $\overline{G}$, there exists an undominated node in $G(H)$, therefore $U(H)$ is a nonempty subset of H. Since G is a graph, for every $h \in U(H)$ there exists a unique maximal partition of $H \setminus \{h\}$ into connected subsets of $\overline{G}$. Also because $\overline{G}$ is a graph, any union of nonempty subsets of at least two sets in such a maximal partition is not connected in G and is therefore not an element of the set system $\mathcal{H}$. Consequently, condition (Q2) is also fulfilled. Condition (Q3) immediately follows since two connected subsets H_1, H_2 of G satisfying $H_1 \cap H_2 \neq \emptyset$ implies that $H_1 \cup H_2$ is connected in $\overline{G}$. $\square$

For a connected graph G, $\mathcal{Q}(\mathcal{G})$ is called the graphical quasi-building set corresponding to G.

Example 5.3 Consider a graph $G = ([n], E)$ with $n = 3$ and $E = A = \{a_1, a_2, a_3\}$ where $a_1 = (1,2), a_2 = (2,3)$, and $a_3 = (3,1)$, i.e., G is a directed circle. The set system underlying the graphical quasi-building set $\mathcal{Q}(\mathcal{G}) = (\mathcal{H}, U)$ corresponding to G is equal to $\mathcal{H} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Because every node is an undominated node in $\{1,2,3\}$, $U(\{1,2,3\}) = \{1,2,3\}$. For each doubleton there is only one node which is undominated in it, $U(\{1,2\}) = \{1\}$, $U(\{1,3\}) = \{3\}$, and $U(\{2,3\}) = \{2\}$. By definition, $U(\{i\}) = \{i\}$ for any $i = 1,2,3$. This $\mathcal{Q}(\mathcal{G})$ has three maximal strictly nested sets, namely

$\mathcal{N}_1 = \{\{1\}, \{1,2\}, \{1,2,3\}\}$, $\mathcal{N}_2 = \{\{2\}, \{2,3\}, \{1,2,3\}\}$, and $\mathcal{N}_3 = \{\{3\}, \{1,3\}, \{1,2,3\}\}$. Notice that this graph has the same set system underlying its graphical quasi-building set as the complete graph does. For the complete graph, however, all vertices in any subset are undominated in it and its graphical quasi-building set has six maximal strictly nested sets.

The example stresses the fact that different connected mixed graphs may have the same set system as a collection of connected subsets. The differences in dominance between vertices within the graphs is expressed in the choice function by the property that the sets of undominated vertices may differ.

Example 5.4 Reconsider Example 3.7, the following quasi-building set $\mathcal{Q}(\mathcal{G}) = (\mathcal{H}, U)$ on $[n] = 2$ where $\mathcal{H} = \{\{1\}, \{2\}, \{1,2\}\}$ and $U(\{1\}) = \{1\},\ U(\{2\}) = \{2\},\ U(\{1,2\}) = \{1\}$. This is the quasi-building set corresponding to the directed graph $G = (V, E)$ with $n = 2$ and $E = A = \{(1,2)\}$.

This example shows that $\mathcal{Q}(\mathcal{G})$ constructed from Definition 5.1 may contain inessential coalitions (in this example, $\{1\}$ is an inessential coalition). It is instead possible to construct a quasi-building set which contains no inessential coalitions.

Definition 5.5 Given a mixed graph $G = (V, E)$ with $E = L \cup A$ on $[n]$, the pair $\mathcal{Q}(\mathcal{G}) = (\mathcal{H}, U)$ on $[n]$ consists of a set system $\mathcal{H}$ and mapping $U : H \to 2^{[n]}$ given by the following conditions:

- $\mathcal{H}$ is the set of all subsets of $[n]$ where each of them is connected in $\overline{G}$ and there is no directed link emanating from such a subset to vertices outside of it.
- U assigns to every $H \in \mathcal{H}$ a set of nodes which are undominated in the subgraph $G(H)$ of G on H, being the set of nodes in H from which there is a directed path in the subgraph $G(H)$ to any of its predecessors in the set H.

This $\mathcal{Q}(\mathcal{G})$ contains no inessential coalitions. As we see in the previous section, the AMV-value of a quasi-building set game does not change with or without inessential coalitions, but the core of the game does.

On the class of directed graph, the average covering tree value is introduced in [16] as the average of marginal contribution vectors corresponds to the set of covering trees induced from a graph. The AMV-value coincides with the average covering tree solution since the set of trees induced from the collection of maximal strictly nested sets on a graphical quasi-building set coincides with the collection of covering trees of the underlying graph, and given a tree the marginal contribution vector is calculated in the same way.

5.2 Set system quasi-building set

In this subsection we discuss quasi-building sets that can be constructed to extract the collection of all maximal strictly nested sets when set systems like augmenting systems, posets and convex geometries are the underlying set system.

Definition 5.6 Given a set system $\mathcal{F}$ on $[n]$, the pair $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$ consists of the set system $\mathcal{H}$ and the mapping $U : \mathcal{H} \to 2^{[n]}$ given by the following conditions:

- $\mathcal{H} = \mathcal{F}$.
- $U(H) = \{i \in H \mid \text{there exists a unique maximal partition of } H \setminus \{i\} \text{ satisfying (Q2)}\}$.

The pair $\mathcal{Q}(\mathcal{F})$ consists of the original set system and the choice function is such that for each feasible coalition, all players who can leave the coalition while satisfying the consistency condition (Q2) can be chosen.

One of the concepts of set systems is augmenting systems, introduced in [4]. A set system $\mathcal{F}$ on $[n]$ is an augmenting system on $[n]$ if it satisfies the following conditions:

(S1) $\emptyset \in \mathcal{F}$.

(S2) If $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{F}$.

(S3) If $S, T \in \mathcal{F}$ and $S \subset T$, then there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

An augmenting system is a set system satisfying (weak) union closedness (S2) and one-point extension (S3). Note that a set system corresponding to an undirected graph is an augmenting system. Antimatroids, another class of set systems, introduced in [10], is a subclass of the class of augmenting systems. In [13], a restriction among players is expressed as a precedence constraint, which is represented by a poset. Given a poset, feasible coalitions are defined as the collection of ideals of the poset, which belongs to a subclass of antimatroids, see [2].

Lemma 5.7 *For an augmenting system $\mathcal{F}$ on $[n]$ with $[n] \in \mathcal{F}$, $\mathcal{Q}(\mathcal{F})$ is a weakly union-closed quasi-building set on $[n]$.*

Proof. Let $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$. Condition (Q1) is satisfied by assumption and condition (S1). Condition (S2) implies condition (Q3). By definition $U(H) \subset H$. Regarding the non-emptiness of the set $U(H)$, it follows from conditions (S1) and (S3) with $S = \emptyset$, that for any $H \in \mathcal{H}$, there exists at least $h \in H$ such that $H \setminus \{h\} \in \mathcal{H}$. □

The notion of convex geometry is introduced in [11]. Set system $\mathcal{F}$ is a convex geometry on $[n]$ if it satisfies the following conditions:

(C1) $\varnothing \in \mathcal{F}$.

(C2) If $S, T \in \mathcal{F}$ then $S \cap T \in \mathcal{F}$.

(C3) If $S \in \mathcal{F}, S \neq [n]$, then there exists $i \in [n] \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

A convex geometry is a set system satisfying intersection closedness (C2) and another form of one-point extension (C3). Note that a graph may not be expressed as a convex geometry, since the collection of connected subgraphs of a graph may not satisfy intersection closedness. Different from augmenting systems, the grand coalition $[n]$ is necessarily feasible by definition. Since union closedness does not imply intersection closedness and vice versa, there is no inclusion relationship between the class of convex geometries and the class of augmenting systems. We show in the following that if a quasi-building set is a convex geometry, then it is an intersection-closed quasi-building set. First we show that it is a quasi-building set.

Lemma 5.8 *For a convex geometry $\mathcal{F}$ on $[n]$, $\mathcal{Q}(\mathcal{F})$ is a quasi-building set on $[n]$.*

Proof. Let $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$. Condition (Q1) follows from condition (C1) and the fact that $[n] \in \mathcal{F}$ is implied by condition (C3). For condition (Q2), by definition of $U(H)$ it suffices to show that for any $H \in \mathcal{F}$ there exists $i \in H$ such that a unique maximal partition $P(H \setminus \{i\})$ exists. First we show that for any H there exists $h \in H$ such that $H \setminus \{h\} \in \mathcal{H}$. Suppose there is no $h \in H$ such that $H \setminus \{h\} \in \mathcal{F}$. Since $\varnothing \in \mathcal{F}$, H cannot be a singleton and therefore $|H| \geq 2$. From (C3) and $\varnothing \in \mathcal{F}$, there is a sequence of n elements $S_1, \ldots, S_n$, with $|S_k| = k$, $S_k \in \mathcal{H}$, $k = 1, \ldots, n$ and $S_1 \subset S_2 \subset \cdots \subset S_n = [n]$. Consider S_{n-1} and denote it as $[n] \setminus \{i_1\}$. Then from (C2) it follows that $i_1 \notin H$, otherwise $H \cap ([n] \setminus \{i_1\})$ will be equal to $H \setminus \{i_1\} \in \mathcal{H}$, which contradicts the supposition. Next, consider S_{n-2} and denote it as $[n] \setminus \{i_1, i_2\}$. Similarly, it holds that $i_2 \notin H$. Now consider $S_{|H|}$ and let $T = [n] \setminus S_{|H|}$. $H \cap T = \varnothing$ and therefore $S_{|H|} = H$. Then $S_{|H|-1} \in \mathcal{F}$ and there exists $h \in H$ such that $S_{|H|-1} = H \setminus \{h\}$, which again is a contradiction. □

From the lemma it follows that the collection of maximal strictly nested sets of such a quasi-building set will contain chains of length n. It also follows that for any feasible coalition there is a chain where the coalition is a part of it. Further, there may exist maximal strictly nested sets which are not a chain. The next lemma concerns the choice function of such a quasi-building set.

Lemma 5.9 *Given a quasi-building set $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$ where $\mathcal{F}$ is a convex geometry on $[n]$, for any two feasible coalitions $T, S \in \mathcal{F}$ with $T \subset S$ it holds that $i \in U(S) \cap T$ implies $i \in U(T)$.*

Proof. Take any $S, T \in \mathcal{F}, T \subset S$ where $P(S \setminus \{i\})$ exists and $i \in T$. From $P(S \setminus \{i\})$, pick a set of feasible coalitions $S_1, \ldots, S_l$ which covers $T \setminus \{i\}$, i.e., $S_k \cap (T \setminus \{i\}) \neq \varnothing$ for any $k = 1, \ldots, l$ and $T \setminus \{i\} \subset \bigcup_{k=1}^{l} S_k$. Then from intersection closedness between T and $S_1, \ldots, S_l$, there is a partition $T_1, \ldots, T_l$ of $T \setminus \{i\}$ where $T_k = T \cap S_k,\ k = 1, \ldots, l$. Further, the set $\{T_1, \ldots, T_l\}$ is the maximal partition of $T \setminus \{i\}$ since $P(S \setminus \{i\})$ is a maximal partition of $S \setminus \{i\}$ satisfying (Q2). Therefore $P(T \setminus \{i\})$ exists and $i \in U(T)$. □

This result shows that condition (Q4) holds for the choice function of the quasi-building set coming from a convex geometry. Condition (Q5) is implied from intersection-closedness of convex geometries and therefore we have the following.

Corollary 5.10 *For a convex geometry $\mathcal{F}$ on $[n]$, $\mathcal{Q}(\mathcal{F})$ is an intersection-closed quasi-building set on $[n]$.*

One of the solutions defined on the class of augmenting systems or convex geometries is the Shapley value in [5] and [7], which considers maximal chains of length n. Given a set system $\mathcal{F}$ which is a convex geometry or an augmenting system, the AMV-value of the quasi-building set game, where the quasi-building set is constructed as

- $\mathcal{H} = \mathcal{F}$,
- $U(H) = \{i \in H \mid H \setminus \{i\} \in \mathcal{F}\}$ for each $H \in \mathcal{F}$,

will coincide with the Shapley value in [5] and [7]. Note that the one-point extension condition of underlying set systems ensures that $U(H) \neq \emptyset$ for all $H \in \mathcal{F}$.

As we show in Section 2, when the set system is a building set the choice set of a feasible coalition defined as Definition 5.6 is the coalition itself. Further, it can be shown that the AMV-value of $\mathcal{Q}(\mathcal{F})$ of a building set $\mathcal{F}$ is equal to the gravity center solution introduced in [17], because the collection of maximal strictly nested sets of a building set $\mathcal{F}$ defined in [17] is identical to the collection of maximal strictly nested sets of $\mathcal{Q}(\mathcal{F})$ and corresponding marginal vectors are calculated in the same way. In [17] it is shown that, for a building set, the gravity center coincides with the Shapley value defined in [12] using the Monge algorithm. Furthermore, if the building set is the collection of all coalitions of players, then the AMV-value of the corresponding quasi-building set game $(v, (2^{[n]}, \mathbf{1}))$ is the Shapley value of the game v, see [20], which follows from the fact shown in [17] that in this case the gravity center solution coincides with the Shapley value. Since the quasi-building set $(2^{[n]}, \mathbf{1})$ is intersection-, and weakly union-closed, and any two distinct subsets of player set is both union-closed and strongly union-closed, both $\mathcal{Q}$-supermodularity and $\mathcal{Q}$-convexity in that case boil down to the usual supermodularity. The quasi-building set $(2^{[n]}, \mathbf{1})$ is also a chain quasi-building set, and $\mathcal{Q}$-chain increasing in that case implies the usual supermodularity (take an increase sequence S^1, S^2, S^3 and H as $S^1 = S \cap T$, $S^2 = S$, $S^3 = S \cup T$ and $H = T$ for any $S, T \in 2^{[n]}$).

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